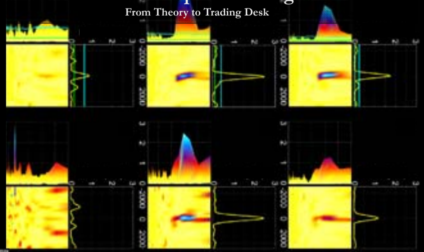
DFT methods on Gaussian Grids for Fast Option Pricing

From Theory to Trading Desk



Marcello Minenna - Paolo Verzella

RISK Europe 07 – June 13, 2007 - London

Syllabus of the presentation

- Option Pricing via DFT
  - FT Pricing formula
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids
- Fast Option Pricing
  - FFT
  - Non Uniform FFT
    - Gaussian Gridding: a matter of interpolation
    - The Computational Framework: Speed, Stability, Accuracy
- Conclusions

Syllabus of the presentation

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FT Pricing Formulas

European Call Price C_t

Spot Price S_t

$$f_2(\ln S_T,\xi\,|\ln S_0)=\int_{-\infty}^{+\infty} e^{i\xi\ln S_T} q_2(\ln S_T\,|\ln S_0)\,d\ln S_T$$

under risk-neutral measure

A linear direct mapping from Fourier Space

FT

FT Pricing Formulas

European Call Price C_t

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under risk-neutral measure

A linear direct mapping from Fourier Space (Carr – Madan)

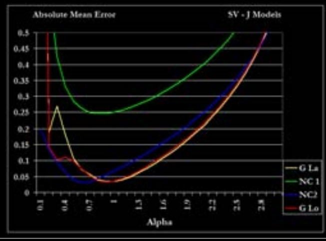
FT

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

FT Pricing Formulas

Accuracy

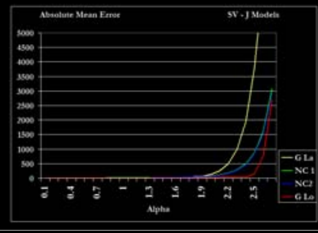
Absolute Mean Error computed w.r.t. α on an (σ,τ) space



FT Pricing Formulas

Stability

Absolute Mean Error computed w.r.t. α on an Extended (σ,τ) space



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  - **DFT Convergence to FT**
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids

DFT Convergence to FT

Given the General DFT

$$\omega(m)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j)\quad where\ m=1,2,\ldots M$$

DFT Convergence to FT

Given the General DFT

$$\omega(m)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j)\quad where\ m=1,2,\ldots M$$

$M\neq N$

DFT Convergence to FT

The Convergence Theorem (C Th)

$$\mathcal{F}[f(x)](t_m)=\lim_{N\to\infty}\frac{X}{N}\omega(m)$$

$$t_m=\frac{2\pi}{X}(m-1)$$

DFT Convergence to FT

C_0 via FT

Convergence theorems

$\overrightarrow{C_0}$ via DFT

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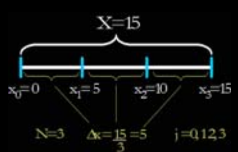
Convergence Theorems for Uniform Grids

Condition 1

Uniform Discretization Grid



Convergence Theorems for Uniform Grids

Condition 2

N=M

DFT specialized

$$\omega(n)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(n-1)}f(x_j)\quad where\ n=1,2,\ldots,N$$

Convergence Theorems for Uniform Grids

Condition 1 Condition 2

DFT Simplified Formula

$$\omega(n)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-1)}f(x_j)\quad where\ n=1,2,\ldots N$$

Nyquist – Shannon Limit (N-S)

$$\mathcal{F}[f(x)](t_n) = \lim_{N\to\infty} \frac{X}{N}\omega(n)$$

$\{t_n\}_{n=1...\frac{N}{2}}$ *for N even*

$\{t_n\}_{n=1...\frac{N+1}{2}}$ *for N odd*

$$C_0 = \frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

Uniform Discretization Grids for *f*

1. $f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\psi_0((j-1)\eta)$
2. $f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\psi_0((j-1)\eta)\cdot\left(3+(-1)^j-\delta_{j-1}\right)$

1.

$$C_0 = \frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\psi_0((j-1)\eta)$$

\+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_T - b + \lambda(u-1)]}}{b}\cdot\Re(\omega(u))$$

2.

$$C_0 = \frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\psi_0((j-1)\eta)\cdot\left(3+(-1)^j-\delta_{j-1}\right)$$

\+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_T - b + \lambda(u-1)]}}{3b}\cdot\Re(\omega(u))$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Trapezoid/Simpson Quadrature Rule

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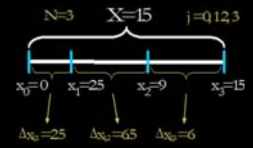
Condition 1

Non Uniform Discretization Grid



Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre

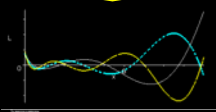
Gander Gautschi

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre



Zeros of Laguerre Poynomials

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of Legendre Poynomials

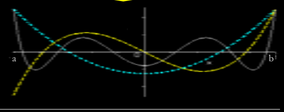
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gander Gautschi

Zeros of rescaled Legendre Poynomials



Condition 2

N≠M

General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)}f(x_j)$$ *where m=1,2,…,M*

The Convergence Theorem (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N\to\infty} \frac{X}{N}\omega(m)$$

$$t_m = \frac{2\pi}{X}(m-1)$$

$$C_0 = \frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

Gaussian Grids for *f*

1. $f(v_{j-1}) = e^{\left[1+i\left(\frac{M\tau}{a}-\ln S_t\right)\right]v_{j-1}}\psi_0(v_{j-1})\ \frac{1}{L_{N+1}(v_{j-1})L_N'(v_{j-1})}$
2. $f(\frac{1}{2}a(1+v_{j-1})) = e^{\left[1+i\left(\frac{M\tau}{a}-\ln S_t\right)\right]\frac{1}{2}a(1+v_{j-1})}\psi_0(\frac{1}{2}a(1+v_{j-1}))\ \frac{1}{[P_{N-1}(v_{j-1})]^2}$

1.

$$C_0 = \frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1}) = e^{\left[1+i\left(\frac{M\tau}{a}-\ln S_t\right)\right]v_{j-1}}\psi_0(v_{j-1})\ \frac{1}{L_{N+1}(v_{j-1})L_N'(v_{j-1})}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx -\Re\left[\frac{e^{-\alpha\left(\ln S_T-\frac{M\tau}{a}+\frac{2\pi}{a}(u-1)\right)}}{\pi}\frac{1}{N+1}\cdot\omega^*(u)\right]$$

2.

$$C_0 = \frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(\tfrac{1}{2}a(1+v_{j-1})) = e^{\left[1+i\left(\frac{M\tau}{a}-\ln S_t\right)\right]\frac{1}{2}a(1+v_{j-1})}\psi_0(\tfrac{1}{2}a(1+v_{j-1}))\ \frac{1}{[P_{N-1}(v_{j-1})]^2}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx \Re\left[\frac{e^{-\alpha\left(\ln S_T-\frac{M\tau}{a}+\frac{2\pi}{a}(u-1)\right)}}{\pi}\frac{1}{N(N-1)}\cdot\omega^*(\tfrac{1}{2}a(1+v_{j-1}))\right]$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Gauss Laguerre/Gander Gautschi Quadrature Rule

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$\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms

$\overrightarrow{C_t}$ via DFT

Newton-Cotes

Uniform FFT

$\overrightarrow{C_t}$ via DFT

Gauss

NonUniform FFT

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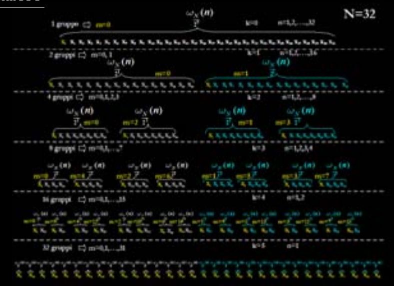
Cooley-Tukey DFT Characterization

$$\omega_2^m(n) = f(x_m) + W_2^{(n-1)} f(x_{m+N/2}) \quad for\ n = 1,2$$

Iterated Bottom – Up for N stages

It gives the FFT Cooley – Tukey Algorithm

N=32



FFT Cooley – Tukey Algorithm

The DFT computational cost drops

from

$O(N^2)$

to

$O(N \log_2 N)$

Since the Nyquist – Shannon Limit, the pricing formulas

via FFT

Give accurate prices
ONLY
Around the Nyquist Frequency

Since the Nyquist – Shannon Limit, the pricing formulas

via FFT

Give accurate prices
ONLY
Around the Nyquist Frequency

Approx. 25% of prices can be accepted

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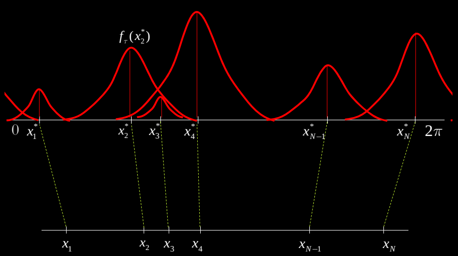
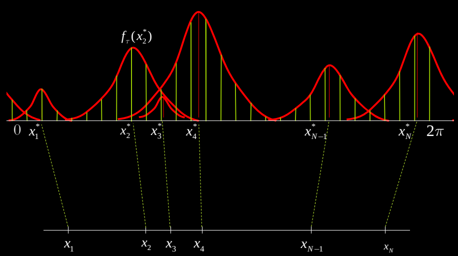
Gaussian Gridding

Step 1

Gaussian Projection of the non uniformly sampled characteristic function on a oversampled uniform grid

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$





Gaussian Gridding

Step 2

FFT computation on the oversampled grid of the Fourier Coefficient of the reprojected characteristic function

$$F_\tau(n) = \frac{1}{2\pi}\int_0^{2\pi} f_\tau(x) e^{-ix(n-1)} dx$$

Gaussian Gridding

Step 3

Elimination of frequencies greater than Nyquist – Shannon Limit

Gaussian Gridding

Step 4

homothetic rescaling from Gaussian scale

$$\omega(n) = \sqrt{\frac{\pi}{\tau}}\, e^{n^2\tau} F_\tau(n)$$

Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio

$$M_\tau = 2M$$

The total cost of the procedure is $\approx 2M \log 2M$

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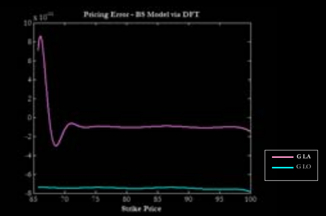
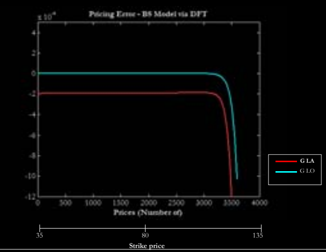
ACCURACY



2000 Prices computed

STABILITY



STABILITY

The error of 90% of prices computed lies in the

10^{-3}

RANGE OF PRECISION

SPEED

the NU – FFT is around 2 time slower than FFT

SPEED

At very low time scales, the differences disappear

	NC2	G – LA	G – LO
FFT	0.01 sec.	N/A	N/A
NU – FFT	0.02 sec.	0.0261 sec.	0.0301 sec.

Computation of 4000 prices on a Centrino 1600Mhz – 2gb RAM
Mean Value over 1000 runs

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- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit
- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT
- NU – FFT speed performances are indistinguishable from FFT's ones

NU – FFT
is a natural candidate for operational use on trading desks